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Existence and stability of stationary vortices in a uniform shear flow

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Isolated vortices in a background flow of constant shear are studied. The flow is governed by the two-dimensional Euler equation. An infinite family of integral invariants, the Casimirs, constrain the flow to an isovortical surface. An isovortical surface consists of all flows that can be obtained by some incompressible deformation of a given vorticity field. It is proved that on every isovortical surface satisfying appropriate conditions there exists a stationary solution, stable to all exponentially growing disturbances, which represents a localized vortex that is elongated in the direction of the external flow. The most important condition is that the vorticity anomaly q in the vortex has the same sign as the external shear. The validity of the proof also requires that q is non-zero only in a finite region, and that $0 < q_{min} \leqslant q \leqslant q_{max} < \infty$ in this region (assuming the external shear to be positive).

1. Introduction

Coherent vortices are often observed in natural shear flows. Their vorticity usually has the same sign as that of the background flow, and they are elongated in the direction of this flow. One example is the Great Red Spot of Jupiter, which extends more than 20000 km in the longitudinal direction, and has existed for at least 300 years. There also exist many other coherent vortices on Jupiter and the other large planets. In most cases they are anticyclones situated in regions of anticyclonic shear of the zonal flow, and their longitudinal extent (in the direction of the external flow) is typically about twice as long as the latitudinal extent. Similar vortices also appear spontaneously in shear flows in both laboratory experiments (Antipov *et al.* 1985; Sommeria, Meyers & Swinney 1988) and numerical simulations (Marcus 1990; Toh, Ohkitani & Yamada 1991).

Theoretically, stationary vortex solutions with opposite signs of the vorticity anomaly and the external shear can also be found. Such counter-rotating vortices are rarely seen in nature, perhaps because they are unstable. In numerical simulations it is observed that they are sheared away and stretched out to long filaments by the external flow, while vortices rotating in the same direction as the outer shear flow are much more robust (Marcus 1990; Toh *et al.* 1991).

The theoretical understanding of this difference mainly comes from analysing elliptic vortex patches with constant vorticity, which are exact stationary solutions of the two-dimensional Euler equation with a uniform external shear (Moore & Saffman 1971). If the shear and the vorticity anomaly have the same sign, such vortex patches are linearly stable. If they have opposite signs, stationary solutions only exist if the vorticity of the external shear is weaker than 0.21 times the vorticity anomaly of the vortex. There are then two stationary solutions with different ellipticity. The more

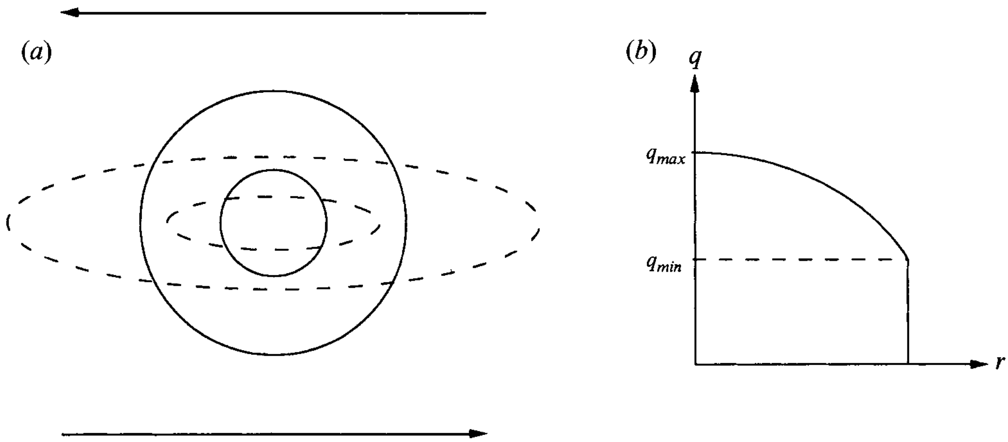

FIGURE 1. (a) A circular vortex is placed in a uniform shear flow. The circles are contour curves of vorticity, and the dashed curves are the same contour curves after the vortex has been deformed into a stationary solution. The area inside any contour curve is preserved during the deformation. The proof presented in this work guarantees that a stationary solution can be constructed by such a deformation. (b) General shape of the radial vorticity profiles for which the proof is valid. q must have finite support (defined as the region where $q \neq 0$) and satisfy $0 < q_{min} \leqslant q \leqslant q_{max} < \infty$ on its support.

elongated one is unstable, while the less elongated one is linearly stable. However, the simulations by Marcus (1990) show that even vortices of the latter kind are easily destroyed by finite-amplitude perturbations.

In this paper, we study stationary vortices in a uniform external shear flow without any restriction to elliptical shape or piecewise-constant vorticity. The problem we solve is illustrated in figure 1, and may be described as follows. Suppose that a circular vortex with monotonic radial vorticity profile is placed in a uniform shear flow. Can a stationary vortex be obtained by deforming this vortex incompressibly, so that the vorticity in each fluid element is conserved during the deformation? (The flows that can be obtained by incompressible deformations of the vorticity field are said to lie on the same 'isovortical surface', or 'symplectic leaf'.) It will be shown that if the vorticity of the external shear and the circular vortex have the same sign, the answer is in general yes. (For the proof to be valid, some additional conditions on the vorticity distribution, which are listed at the beginning of §5 and illustrated in figure 1(b), must also be satisfied.) Moreover, the stationary solution is obtained by that incompressible deformation which maximizes the energy of the flow, and it is therefore stable.

This does not mean, of course, that the flow will evolve toward this stationary solution if a circular vortex in uniform shear flow is taken as initial condition. Indeed, this stationary state is inaccessible, since its energy is larger than that of the initial state. In the simplest case, with a circular patch of constant vorticity as initial condition, the vortex will perform regular oscillations around the elliptical stationary state of maximum energy, as demonstrated by Kida's (1981) exact solutions. With a non-constant vorticity profile the time evolution will presumably be more irregular, but qualitatively similar.

In more mathematical terms, the following will be proved: on every isovortical surface such that the external shear and the vorticity anomaly have the same sign (and some additional technical conditions are satisfied), there exists a maximum energy flow. This flow contains a localized, stationary and spectrally stable vortex. The method of the proof is similar to that of Benjamin (1976), who proved that three-dimensional vortex rings maximize the energy on isovortical surfaces (restricting the dynamics to axisymmetric perturbations), and Burton (1988), who proved the existence of nonlinear dipole vortex solutions.

In §2 the problem is stated, and a simple intuitive argument for the existence of a maximum energy flow is given. A rigorous proof is presented in §3, which is more mathematical than usual for a paper in fluid dynamics. (Many readers will probably be satisfied with the simple argument in §2.) The implications for stability are studied in §4, and in §5 the results are summarized and discussed.

2. Formulation of the problem and heuristic solution

Assume that the background flow is given by $V = -Sy\hat{x}$, where S is the strength of the shear. The Euler equation for two-dimensional incompressible flow can then be written

$$\frac{\partial q}{\partial t} + \{\Psi, q\} = 0, \tag{1}$$

where $\{,\}$ denotes the Poisson bracket (or Jacobian), $q = \nabla^2\phi$ is the vorticity anomaly, and $\Psi \equiv \frac{1}{2}Sy^2 + \phi$ is the streamfunction of the total flow $V + v$, where $v = \hat{z} \times \nabla\phi$. We assume that q has compact support. (The support of q is defined as the region where $q \neq 0$, and compact means closed and bounded.) Equation (1) then conserves the infinite family of Casimir functionals,

$$C_F = \int F(q)\,\mathrm{d}\boldsymbol{r}, \tag{2}$$

where F is an arbitrary function and the integral is taken over the (x, y)-plane. It also conserves the energy,

$$\begin{aligned} E &= -\frac{1}{2}\int (Sy^2 + \phi)\, q\,\mathrm{d}\boldsymbol{r} \\ &= -\int \frac{Sy^2}{2} q\,\mathrm{d}\boldsymbol{r} - \frac{1}{4\pi}\iint q(\boldsymbol{r}_1)\, q(\boldsymbol{r}_2) \ln(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2. \end{aligned} \tag{3}$$

The total energy of the vortex is actually infinite, since the circulation is non-zero, but in (3) this singularity has been removed.

We now want to maximize the energy, while keeping all Casimir functionals fixed. This means that we are constrained to a particular 'symplectic leaf', or 'isovortical surface'. (In mathematical terminology, all functions $q(\boldsymbol{r})$ on the same isovortical surface are said to be 'rearrangements' of one another.) An isovortical surface is specified by the function

$$A_q(\mu) = \int H(q - \mu)\,\mathrm{d}\boldsymbol{r}, \tag{4}$$

where H is the Heaviside function. Thus, $A_q(\mu)$ is the area of the region where $q \geqslant \mu$. It decreases monotonically to zero at $\mu = q_{max}$, and approaches the area of the support of q when $\mu \to 0$.

Imagine that a fixed value of q is assigned to each fluid element in the incompressible fluid. Regardless of how the fluid elements are redistributed over the (x, y)-plane (or 'rearranged'), we are then on the same isovortical surface. If we define $U \equiv -E$, and interpret q as mass density, we see from (3) that U has the form of potential energy due to two-dimensional gravitational forces. The first term represents the contribution from an external gravitational field, and the second one is the interaction energy between the mass elements. If $S = 0$ and $q \geqslant 0$ everywhere, the minimum potential energy (i.e. the maximum of E) is of course attained by a circularly symmetric distribution with the heaviest matter at the centre. (Proof is given below.)

If $S > 0$, this vortex (or 'lump of matter') is placed in an external potential valley with the shape $Sy^2/2$. It seems intuitively obvious that the minimum potential energy (or maximum of E) is attained when it lies at the bottom of the valley. (A rigorous proof is given in the next section. It will be seen that the existence of a minimum is in fact not so obvious, and depends crucially on the long-range nature of the Green's function.) It will be somewhat squeezed together by the external field, so that the shape is no longer circular, but the vorticity anomaly (or 'density') still decreases monotonically outward from the centre. Any incompressible deformation (re-arrangement) of this configuration will decrease E, with the exception of a rigid translation in the x-direction, which of course leaves it unchanged.

To first order, a general isovortical perturbation (i.e. a perturbation that does not affect the Casimirs) is given by $\delta q = \delta \boldsymbol{r} \cdot \boldsymbol{\nabla} q$, where $\boldsymbol{\nabla} \cdot \delta \boldsymbol{r} = 0$. Thus, we can write $\delta q = \{\alpha, q\}$, where $\alpha(x, y)$ is an arbitrary function. To higher orders we have (Benjamin 1976)

$$\Delta q = \delta q + \delta^2 q + \ldots = \{\alpha, q\} + \tfrac{1}{2}\{\alpha, \{\alpha, q\}\} + \ldots . \tag{5}$$

This expansion can be obtained by direct calculation, imposing the condition that the variation of all Casimirs must vanish to all orders. It can also be obtained as a Lie series in the Hamiltonian formulation of the Euler equation.

For the maximum energy configuration, we must have $\delta E = 0$. Using the first-order term in (5) this implies

$$\{\Psi, q\} = 0. \tag{6}$$

This is the equation for a stationary flow, with the solution $\Psi = \Psi_0(q_0)$. Thus, we have established the existence of a stationary solution on every isovortical surface (i.e. for any prescription of the integrals C_F) such that S and q are positive. (In the rigorous proof below, the additional assumption (10) is also needed.) This solution is a vortex elongated in the direction of the external flow.

The exact shape depends on the integrals C_F. For a patch of constant vorticity it is elliptic, but in general the shape cannot be determined analytically. Numerically it should be easy to find the solution with the relaxation procedure proposed by Carnevale & Vallis (1990). With their algorithm the vorticity is advected by an artificial incompressible velocity field. This field is chosen so that the energy increases monotonically, while all the Casimirs are automatically conserved, until a steady state is reached.

A general measure of the ellipticity of a stationary solution can be obtained by multiplying (6) by xy and integrating. After partial integration we obtain

$$\int \left[\left(\frac{\partial \phi}{\partial y} \right)^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 \right] \mathrm{d}\boldsymbol{r} = S \int y^2 q \, \mathrm{d}\boldsymbol{r},$$

where we assumed that $\boldsymbol{\nabla}\phi$ decreases as $1/r$ at infinity, so that the boundary terms vanished. The left-hand side of this relation gives a measure of the elongation in the x-direction. It is positive if S and q have the same sign.

3. Proof of the existence of an energy maximum

In this section we will prove rigorously that there exists a maximum energy flow on every isovortical surface that satisfies the appropriate conditions. We then need theorems on symmetrization, which is a particular kind of rearrangement. Given a function of one variable $q(x)$, the symmetrized function $q^*(x)$ is even, and non-increasing for $x > 0$. For functions $q(\boldsymbol{r})$ of several variables, the symmetrized function is a non-increasing function of $r = |\boldsymbol{r}|$. We have the following formal definition.

Definition: Let $D_\mu = \{\boldsymbol{r} \in D : q(\boldsymbol{r}) \geqslant \mu\}$, and let D_μ^* be the corresponding symmetrized domain, i.e. the sphere $r \leqslant \rho$ or $r < \rho$ with the same volume as D_μ. Then, for any given $\boldsymbol{r} \in D^*$,

$$q^*(\boldsymbol{r}) = \sup\{\mu : \boldsymbol{r} \in D_\mu^*\}.$$

Here D is the domain of definition of q; q^* is said to be symmetrically decreasing. Since q^* is a rearrangement of q, it of course satisfies the Casimir constraints: $C_F[q] = C_F[q^*]$ for arbitrary F, where the integral is taken over D and D^*, respectively. Moreover, if $q(\boldsymbol{r})$ is Lipschitz continuous, then so is $q^*(\boldsymbol{r})$ (Bandle 1980).

We will need the following inequality (theorem 380 in Hardy, Littlewood & Polya 1952):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)\,q(y)\,g(x-y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p^*(x)\,q^*(y)\,g(x-y)\,\mathrm{d}x\,\mathrm{d}y, \qquad (7)$$

where p and q are non-negative functions, and g is symmetrically decreasing. By symmetrizing infinitely many times along various directions, (7) can be generalized to functions of several variables (Sobolev 1963):

$$\iint p(\boldsymbol{r}_1)\,q(\boldsymbol{r}_2)\,g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2 \leqslant \iint p^*(\boldsymbol{r}_1)\,q^*(\boldsymbol{r}_2)\,g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2. \qquad (8)$$

If $S = 0$, (8) can immediately be applied to the energy E in (3). Identifying g with the Green's function, and setting $p \equiv q$, we find that the energy is maximum for $q^*(\boldsymbol{r})$, i.e. for a circular vortex with monotonic decreasing vorticity profile.

If $S > 0$ the energy is still bounded from above, since the first term in (3) is negative definite, but we cannot find the maximum explicitly. We may symmetrize along the x-axis, which according to (7) increases the second term in (3), while leaving the first one unchanged, and also along the y-axis, which increases both terms. (For the first term we then use the inequality $\int fg\,\mathrm{d}y \leqslant \int f^*g^*\,\mathrm{d}y$, lemma 2.4 in Bandle 1980.) We conclude that the maximizing function $q(\boldsymbol{r})$ must be symmetric decreasing in both the x- and y-directions. However, we cannot symmetrize in any other direction, since this would change the externally given function y^2.

We will now show that the upper bound of E is attained by some function $q(\boldsymbol{r})$, which is therefore the stationary solution we are looking for. The key concept is compactness, which is absent *a priori* for two reasons. One is that the functions q_i in a maximizing sequence (i.e. a sequence for which $E[q_i]$ approaches the upper bound of E) might become more and more fragmented and rapidly oscillating. The other is that the support of q_i might be stretched out to an ever longer and thinner filament. In neither case would the sequence converge to an acceptable solution. The proof will be done in two steps, dealing with these issues separately. In the first step we confine the problem to the square $D = \{x, y : -L \leqslant x, y \leqslant L\}$, and in the second step we show that this restriction can be removed.

We first observe that the set of bounded functions $|f| \leqslant f_{max}$ on a bounded domain D is the same in $L^1[D]$ as in $L^2[D]$. This follows from the inequalities

$$\|f\|_{L^2}^2 = \int_D |f|^2\,\mathrm{d}\boldsymbol{r} \leqslant f_{max}\int_D |f|\,\mathrm{d}\boldsymbol{r} = f_{max}\,\|f\|_{L^1},$$

and
$$\|f\|_{L^1}^2 = \left(\int_D |f|\,\mathrm{d}\boldsymbol{r}\right)^2 \leqslant \left(\int_D |f|^2\,\mathrm{d}\boldsymbol{r}\right)\left(\int_D \mathrm{d}\boldsymbol{r}\right) = \|f\|_{L^2}^2\,A(D),$$

where $A(D)$ is the area of D, and we used Cauchy–Schwarz inequality. Thus, we can use whichever norm is the most convenient in the estimates below.

Let Σ_D denote the set of all possible rearrangements of some given function $q(\boldsymbol{r})$ that have their support entirely in D. Assume that $0 \leqslant q \leqslant q_{max}$. Consider a maximizing sequence $\{q_i\}$, where $q_i \in \Sigma_D$. As explained above, we may assume that all the functions q_i are symmetric decreasing in x and y. In Appendix A it is proved that the set of all functions f on D that are symmetric decreasing in x and y and satisfy $0 \leqslant f(\boldsymbol{r}) \leqslant f_{max}$ is totally bounded in $L^2[D]$ and $L^1[D]$. (The definition of 'totally bounded' is given in Appendix A. A set which is totally bounded and complete is compact, which guarantees that every sequence has a convergent subsequence.) Since the sequence $\{q_i\}$ is a subset of a totally bounded set it is totally bounded, hence it contains a Cauchy sequence. This Cauchy sequence must converge to some $\hat{q}$ in $L^2[D]$ and $L^1[D]$, since these spaces are complete. Furthermore, since E is a continuous functional on $L^2[D]$ (as shown in Appendix B), $E[\hat{q}]$ is the maximum energy.

It remains to show that $\hat{q} \in \Sigma_D$, i.e. that $\hat{q}$ is a rearrangement of q_i. We do this by showing that the functions $A_{\hat{q}}(\mu)$ and $A_{q_i}(\mu)$, defined in (4), are equal. We have

$$\begin{aligned}\int_0^{q_{max}} |A_{q_i}(\mu) - A_{\hat{q}}(\mu)|\,\mathrm{d}\mu &= \int_0^{q_{max}} \left| \int_D [H(q_i-\mu) - H(\hat{q}-\mu)]\,\mathrm{d}\boldsymbol{r} \right| \mathrm{d}\mu \\ &\leqslant \int_0^{q_{max}} \int_D |H(q_i-\mu) - H(\hat{q}-\mu)|\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\mu \\ &= \int_D \int_0^{q_{max}} |H(q_i-\mu) - H(\hat{q}-\mu)|\,\mathrm{d}\mu\,\mathrm{d}\boldsymbol{r} \\ &= \int_D |q_i - \hat{q}|\,\mathrm{d}\boldsymbol{r} \rightarrow 0, \quad i \rightarrow \infty,\end{aligned}$$

since, as already shown, $q_i \rightarrow \hat{q}$ in L^1-norm. Since all the functions $A_{q_i}(\mu)$ are identical, this shows that $A_{\hat{q}}(\mu) = A_{q_i}(\mu)$ for all μ, except perhaps on a set of measure zero. But from the definition (4) we know that both these functions are monotonic decreasing, and attain the left limit (i.e. the largest value) at any discontinuity point. It is then clear that in fact $A_{\hat{q}}(\mu) = A_{q_i}(\mu)$ for all μ. This completes the first part of the proof.

The maximizer $\hat{q}$ must be symmetric decreasing in x and y, and in the interior of D it must satisfy (6). This implies that if $\hat{q}$ is discontinuous in one point, it must also be discontinuous everywhere on the isoline of Ψ where this point is located. Hence, $\hat{q}$ can only be discontinuous for the same values of q as the symmetrized function $q^*(r)$. Loosely speaking, $\hat{q}$ is as smooth as the smoothest functions in Σ_D.

The second step is to show that if L is large enough, then the support of $\hat{q}$ is everywhere at a finite distance from the boundary of D. Then, letting $d > 0$ denote the minimum distance between the support of $\hat{q}$ and the boundary of D, and $L_{\hat{q}} \equiv L - d$ the maximum half-diameter of the support (cf. figure 2), $\hat{q}$ and $L_{\hat{q}}$ (and the energy of $\hat{q}$) do not change if L is decreased by an amount smaller than d. If L is decreased more than this, the support touches the boundary of D, i.e. $L_{\hat{q}} = L$. Hence, there exists some critical value L_c such that $L_{\hat{q}} = L$ for $L < L_c$ and $L_{\hat{q}} = L_c$ for $L > L_c$. When $L > L_c$ the maximizer $\hat{q}$ is independent of L, and in fact maximizes E in Σ, the set of unrestricted rearrangements of $q(\boldsymbol{r})$.

To show this we assume the opposite, i.e. that the support of $\hat{q}$ touches the boundary of D. Since $\hat{q}$ must be symmetric decreasing, this happens in the points $(\pm L, 0)$. (If it

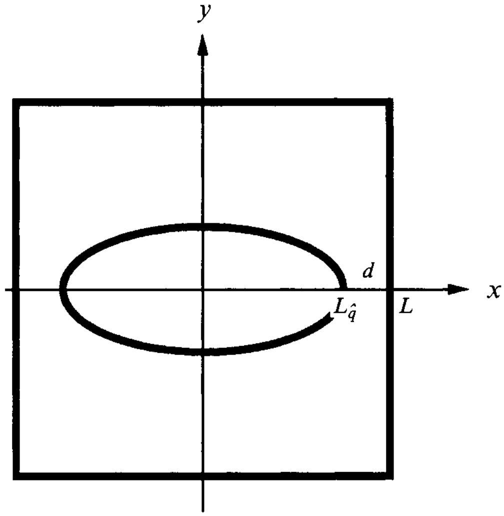

FIGURE 2. Illustration of the energy maximization when the support of q (i.e. the region where $q \neq 0$) is confined to the square $D = \{x, y\colon -L \leqslant x, y \leqslant L\}$. The inner curve is the boundary of the support of the maximizer $\hat{q}$, d denotes the minimum distance between this curve and the boundary of D, and $L_{\hat{q}}$ the distance from the origin to the point where it crosses the x-axis. $L_{\hat{q}}$ is independent of L as long as L is larger than some critical value L_c, and the boundary of D can then be removed without changing the maximizer $\hat{q}$.

happens in the points $(0, \pm L)$ we can use the same procedure as below, but the necessary inequalities are then more easily satisfied, since y_C^2 in (9) is replaced by $y_C^2 - L^2$.) We study the particular rearrangement of $\hat{q}$ by which the infinitesimal circulation dQ is moved from $\boldsymbol{r}_L = (L, 0)$ to a point $\boldsymbol{r}_C = (x_C, y_C)$ as close as possible to the origin, but outside the support of $\hat{q}$. The energy change due to this rearrangement is given by

$$dE = -dQ[\tfrac{1}{2}Sy_C^2 + \phi(\boldsymbol{r}_C) - \phi(\boldsymbol{r}_L)],$$

where $\nabla^2\phi = \hat{q}$. If we can show that

$$\phi(\boldsymbol{r}_L) - \phi(\boldsymbol{r}_C) > \tfrac{1}{2}Sy_C^2, \tag{9}$$

then dE is positive, which contradicts the assumption that $\hat{q}$ is a maximizer. Hence the support of $\hat{q}$ cannot touch the boundary of D, and the proof is complete.

To see that (9) is plausible, we assume that most of the vorticity is concentrated in a small region near the origin, so that $\phi(r) \approx (Q/2\pi)\ln(r)$, where $Q = \int q\, d\boldsymbol{r}$ is the total circulation. r_C can always be chosen smaller than $(A/\pi)^{1/2}$, where A is the area of the support of q, and we find that (9) is satisfied if $L > (A/\pi)^{1/2}\exp(SA/Q)$. While nothing is proved by this crude estimate, it does demonstrate that the result depends crucially on the long-range nature of the Green's function.

To show (9), we make the additional assumption that q satisfies

$$0 < q_{min} \leqslant q \leqslant q_{max} < \infty \tag{10}$$

on its support, for some constants q_{min} and q_{max}. (Thus, q is discontinuous on the boundary of its support.) We then estimate the left-hand side of (9) in two steps. For the first step, we divide the square $-L \leqslant x,\ y \leqslant L$ into three parts, cf. figure 3. S_1 is

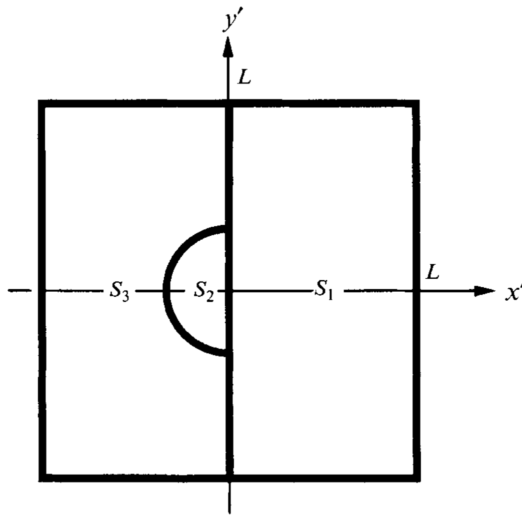

FIGURE 3. Illustration of the domain of integration in (11).

the half-square $x > 0$, S_2 is the half-circle $x < 0$ and $r < (A/\pi^{1/2}$, and S_3 is the remainder of the square. We have

$$\phi(\boldsymbol{r}_L) - \phi(0) = \frac{1}{2\pi}\int_{S_1+S_2+S_3} \hat{q}(\boldsymbol{r}')\ln\left(\frac{|L\hat{x}-\boldsymbol{r}'|}{r'}\right)\mathrm{d}\boldsymbol{r}'. \tag{11}$$

Since $\hat{q}$ is symmetric decreasing in x, the integral over S_1 is positive. Furthermore,

$$\int_{S_2} \hat{q}(\boldsymbol{r}')\ln\left(\frac{|L\hat{x}-\boldsymbol{r}'|}{r'}\right)\mathrm{d}\boldsymbol{r}' > \int_{S_2} \hat{q}(\boldsymbol{r}')\tfrac{1}{2}\ln\left(\frac{L^2\pi}{A}\right)\mathrm{d}\boldsymbol{r}' = \frac{\beta Q}{4}\ln\left(\frac{L^2\pi}{A}\right),$$

where βQ is the circulation of the circle $r < (A/\pi)^{1/2}$, and

$$\int_{S_3} \hat{q}(\boldsymbol{r}')\ln\left(\frac{|L\hat{x}-\boldsymbol{r}'|}{r'}\right)\mathrm{d}\boldsymbol{r}' > \int_{S_3} \hat{q}(\boldsymbol{r}')\ln\left(\frac{\surd 2L}{L}\right)\mathrm{d}\boldsymbol{r}' = \tfrac{1}{4}\ln 2(1-\beta)\,Q,$$

so that

$$\phi(\boldsymbol{r}_L) - \phi(0) > \frac{Q}{8\pi}\left[\beta\ln\left(\frac{L^2\pi}{A}\right) + (1-\beta)\ln 2\right]. \tag{12}$$

The next step is to estimate the maximum difference in ϕ between two close points:

$$\begin{aligned}\phi(0) - \phi(\boldsymbol{r}_C) &= \frac{1}{2\pi}\int \hat{q}(\boldsymbol{r}')\ln\left(\frac{r'}{|\boldsymbol{r}_C-\boldsymbol{r}'|}\right)\mathrm{d}\boldsymbol{r}' > \frac{1}{2\pi}\int \hat{q}(\boldsymbol{r}')\ln\left(\frac{r'}{r'+r_C}\right)\mathrm{d}\boldsymbol{r}' \\ &> -\frac{1}{2\pi}\int q^*(r')\ln\left(1+\frac{r_C}{r'}\right)\mathrm{d}\boldsymbol{r}' > -\frac{q_{max}}{2\pi}\int_0^{(A/\pi)^{1/2}} \ln\left(1+\frac{r_C}{r'}\right)2\pi r'\,\mathrm{d}r' \\ &> -\frac{q_{max}r_C}{2\pi}\int_0^{(A/\pi)^{1/2}} 2\pi\,\mathrm{d}r' = -q_{max}r_C(A/\pi)^{1/2}.\end{aligned} \tag{13}$$

Adding (12) and (13) we obtain

$$\phi(\boldsymbol{r}_L) - \phi(\boldsymbol{r}_C) > \frac{Q}{8\pi}\left[\beta\ln\left(\frac{L^2\pi}{A}\right) + (1-\beta)\ln 2\right] - q_{max}r_C\left(\frac{A}{\pi}\right)^{1/2}. \tag{14}$$

Notice that if $\beta \to 0$, the difference $\phi(\boldsymbol{r}_L)-\phi(\boldsymbol{r}_C)$ may remain bounded as $L\to\infty$. (This is the case, for instance, if the vorticity is stretched out in a narrow strip along the x-axis.) To show (9), we therefore need an estimate of r_C (and hence of y_C) that goes to zero as $\beta < 0$. Since the support of $\hat{q}$ closer than $(A/\pi)^{1/2}$ to the origin has an area less than $\beta Q/q_{min}$, we can choose

$$r_C \leqslant \left(\frac{\beta Q}{\pi q_{min}}\right)^{1/2}. \tag{15}$$

Equations (14) and (15) give

$$\phi(\boldsymbol{r}_L)-\phi(\boldsymbol{r}_C)-\tfrac{1}{2}Sy_C^2 > \frac{Q}{8\pi}\ln 2-\frac{q_{max}}{\pi}\left(\frac{AQ\beta}{q_{min}}\right)^{1/2}+\left[\ln\left(\frac{L^2\pi}{2A}\right)-\frac{4S}{q_{min}}\right]\frac{Q\beta}{8\pi}. \tag{16}$$

The right-hand side of (16) is a second degree polynomial in $\beta^{1/2}$. Since β (i.e. the fraction of the total circulation which is closer than $(A/\pi)^{1/2}$ to the origin) is unknown, we use the value at the minimum point of the parabola, and obtain

$$\phi(\boldsymbol{r}_L)-\phi(\boldsymbol{r}_C)-\tfrac{1}{2}Sy_C^2 > \frac{Q}{8\pi}\ln 2-\frac{2Aq_{max}^2}{\pi}\left[q_{min}\ln\left(\frac{L^2\pi}{2A}\right)-4S\right]^{-1}, \tag{17}$$

where the denominator in the last term is assumed to be positive. Thus, if

$$L^2 > \frac{2A}{\pi}\exp\left(\frac{4S}{q_{min}}+\frac{16Aq_{max}^2}{Qq_{min}\ln 2}\right), \tag{18}$$

then (9) is satisfied, which completes the proof.

The existence proofs for vortex rings by Benjamin (1976) and for dipole vortices by Burton (1988) were based on similar ideas as here. However, the mathematics involved in the present proof is simpler. The basic reason is that it was here possible to symmetrize in both the x- and the y-directions. Together with the theorem of Appendix A this provided the compactness property, and it was therefore not necessary to invoke the rather abstract concepts of weak topology.

We end this section with a conjecture. Consider two different isovortical surfaces Σ_1 and Σ_2, represented by the symmetrized functions $q_1^*(r)$ and $q_2^*(r)$, respectively. Suppose that these two functions have the same support, and that $q_2^*(r) \geqslant q_1^*(r)$. It seems likely that increasing the amplitude of the vortex will make it more circular, i.e. that the support of the maximizing function $\hat{q}_2$ in Σ_2 can be inscribed in a smaller square than the corresponding maximizer $\hat{q}_1$ in Σ_1. (If q_1^* and q_2^* are proportional, this is certainly true.) If this conjecture could be proven, the estimate (18) could be sharpened drastically, and q_{min} could be removed from (10).

4. Stability

The fact that a flow maximizes E on an isovortical surface clearly has implications for the nonlinear stability. If the initial perturbation is isovortical, the system must remain close to the maximum on the isovortical surface. If the perturbation contains some vorticity, the system is initially displaced to some neighbouring isovortical surface. It is then equivalent to a system which is isovortically perturbed from the nearby maximum energy point on this new surface. As argued by Benjamin (1976), this should guarantee stability in a practical sense. However, it is not so easy to formalize this to a statement of stability in some norm.

Spectral stability (i.e. the absence of unstable linear eigenmodes), on the other hand, can easily be shown. For a maximum energy flow, the second-order variation $\delta^2 E$ is negative definite. Hence, using (5),

$$\delta^2 E = \frac{1}{2}\iint\left[(\delta q)^2 \frac{\mathrm{d}\Psi_0}{\mathrm{d}q_0} - \delta\phi\,\delta q\right]\mathrm{d}x\,\mathrm{d}y \leqslant 0 \tag{19}$$

for all variations of the form $\nabla^2\delta\phi = \delta q = \{\alpha, q_0\}$. The equality here holds only for the translation mode, $\alpha = ky$.

The linear equation for small perturbations of a stationary flow, obtained by setting $q = q_0 + q_1$ in (1) and linearizing, conserves the integrals

$$c_f = \int f(q_0)\, q_1\, \mathrm{d}x\,\mathrm{d}y, \tag{20}$$

where f is arbitrary. This is the first-order perturbation of the Casimirs (2), with $f = F'$. Another invariant of the linearized equation is $\delta^2 E$ in (19), with δq replaced by $q_1(t)$. This invariant is usually called the 'pseudoenergy' or the 'Arnol'd invariant' (McIntyre & Shepherd 1987). The corresponding invariant in a plasma physical context is called 'free energy' by Morrison & Pfirsch (1989) and Morrison & Kotschenreuther (1990). If it were positive definite and bounded away from zero for arbitrary perturbations δq (and not just for isovortical ones), the flow would be 'formally stable'.

For a growing eigenmode $\tilde{q}(\boldsymbol{r})\exp[(\gamma - \mathrm{i}\omega)t]$, the value of all invariants must be zero. $c_f = 0$ implies that there exists some function $\alpha(\boldsymbol{r})$ such that $\tilde{q} = \{\alpha, q_0\}$. But from (19) we know that $\delta^2 E$ is negative for all functions $\tilde{q}$ of this form, except for the translation mode, which is obviously not growing. Hence no growing eigenmode exists. (This does not exclude algebraic instabilities, which are a superposition of steady linear eigenmodes.)

We can also exclude explosive instability, the most dangerous nonlinear instability, which arises because of nonlinear coupling between steady eigenmodes with different sign of $\delta^2 E$ (Morrison & Kotschenreuther 1990). Because of (19), the mode with positive $\delta^2 E$ could not conserve all Casimirs, and hence it could not grow without bound.

5. Discussion

In the previous sections, we have proved the existence of a spectrally stable solution on every isovortical surface satisfying the following conditions: $S \geqslant 0$ and $q \geqslant 0$, q should have compact support, and q should satisfy (10) on its support.

The condition $q_{min} > 0$ in (10) is probably purely technical. A large body of experience with contour dynamics simulations indicates that the limit where the number of contours is large, and thus the vorticity discontinuity small, is well-behaved (Legras & Dritschel 1993). There is therefore no reason to believe that a discontinuity at the boundary of the support is necessary for the existence of a localized solution.

The inequality (18), which guarantees that the support of the maximizing function $\hat{q}$ on a bounded domain lies entirely in the interior of the domain, also gives an upper estimate of the size of the vortex. However, as indicated above, the appearance of q_{min} in this expression is probably artificial. If q_{min} is much smaller than the typical vorticity in the vortex, the estimate given by (18) is therefore probably much too large.

Note also that the long-range nature of the Green's function is crucial for the proof. With a short-range Green's function, for which ϕ is finite at infinity, a separatrix appears in stationary solutions. Outside the separatrix the streamlines $\Psi = \text{const.}$ are open, and q must vanish, and it is therefore obvious that a stationary solution does not exist on isovortical surfaces where the support of q is too large.

The case when q and S are both negative is of course equivalent to both being positive. If, on the other hand, q and S have opposite signs, the first term of (3) corresponds to a potential hill (or, rather, ridge), and we can neither prove existence nor stability of any stationary solution. It is known, however, that exact stationary solutions in the form of elliptic patches of constant vorticity exist in this case, unless the external shear is too strong (Moore & Saffman 1971). These patches are lying on top of the potential ridge, stretched out in the transverse direction to the external flow (i.e. 'hanging down' on both sides of the ridge). The energy is then maximized with respect to the internal interaction (the second term of (3)), but minimized with respect to the external field (the first term). Thus, the total energy has a saddle point. Such vortices are easily disrupted by perturbations of finite amplitude, as seen in the simulations by Marcus (1990).

The present analysis is easily extended to a general external strain flow. Assuming that the vorticity anomaly is positive everywhere, one finds that maximum energy configurations exist whenever the external streamfunction is positive definite. This means that the origin is an elliptic stagnation point of the external flow, with the same sign of the vorticity as the vortex. In all other cases, i.e. when it is a hyperbolic stagnation point or a counter-rotating elliptic stagnation point (with opposite signs of the external vorticity and that in the vortex), the energy is indefinite, and nothing can be proved. Examples of this kind are the Kirchhoff vortex (a steadily rotating elliptic vortex patch), and the V-states found numerically by Deem & Zabusky (1978). These solutions correspond to vortex patches on top of a circular potential hill, since the outer flow is counter-rotating in the reference frame where the solutions are stationary.

The intuitive argument in §2 closely follows the ideas of Petviashvili & Yan'kov (1984) and Filippov & Yan'kov (1986), who used the method to prove existence and stability of localized vortices in various plasma equations. One case is the one-dimensional Vlasov–Poisson equations, which are very similar to the two-dimensional Euler equation, and describe the evolution in phase space (x, v) of a gas of charged particles in a self-consistent electrostatic potential. Petviashvili & Yan'kov studied the existence and stability of 'ion holes' and 'electron holes', which are phase-space vortices with a local depletion of the distribution function. Note, however, that the Green's function has short range in this case because of the Debye screening. As explained above, this means that a maximum energy state does not exist on all isovortical surfaces.

Finally, some comments should be made about the implication of the present result for flows on the β-plane (i.e. with a background gradient of potential vorticity) and flows with non-uniform external shear. In these cases the system supports linear waves (i.e. Rossby waves) with positive energy that conserve all Casimirs, hence no maximum energy state exists. If a vortex interacts strongly with such waves it will quickly be disrupted. (Such a process is characteristic for maximum energy states. It is not possible for states that are stationary because they minimize the energy, which is more common in various branches of physics.) The main condition for a vortex to be long-lived is therefore that the wave radiation from the vortex is absent or weak. The most powerful kind of radiation is Cerenkov radiation, and the condition for it to be absent is that the vortex should propagate with a velocity that does not coincide with the phase velocity of any linear waves. This condition has been explored for a number of two-dimensional or quasi-two-dimensional fluid models (Nycander 1994).

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Appendix A. Proof that a bounded sequence of symmetric decreasing functions is totally bounded

For simplicity, we here define the domain as $D = \{x, y: 0 \leqslant x, y \leqslant 1\}$, and let I be the set of functions $f \in L^1[D]$ which are non-increasing in x and y, and satisfy $0 \leqslant f \leqslant 1$. (The generalization to symmetric decreasing functions on $-L \leqslant x, y \leqslant L$ is of course immediate.) In this Appendix, the norm used is always the L^1-norm. We will show that the set I is totally bounded. By definition, we then have to show that given $\epsilon > 0$, there exists a finite number of functions $\varphi_i \in I$ such that

$$\|\varphi_i - f\| < \epsilon \tag{A 1}$$

for any $f \in I$ and some function φ_i. This means, loosely speaking, that a finite number of elements in I can approximate the whole set arbitrarily well.

The approximating functions φ_i are constructed as follows. Divide D into N^2 squares with the side $\delta = 1/N$. The squares are denoted $D_{m,n} = \{x, y: (m-1)\delta < x < m\delta, (n-1)\delta < y < n\delta\}$, where $1 \leqslant m, n \leqslant N$, and m and n are integers. Any φ_i is constant in each square $D_{m,n}$, and is only allowed to assume the discrete values $j\delta$, where $0 \leqslant j \leqslant N$, and j is an integer. There are $(N+1)^{N^2}$ such functions. Let the approximating set $\{\varphi_i\}$ be the non-increasing subset of these functions.

Given some function $f \in I$, we first show that it can be approximated by a piecewise-constant function $\bar{f}$. Let $\bar{f}(x, y)$ be equal to the infimum of f over the square $D_{m,n}$ where (x, y) is situated. Since f is non-increasing in x and y, we get

$$\bar{f}(x,y) = \inf_{D_{m,n}} f(x,y) \geqslant \sup_{D_{m+1,n+1}} f(x,y) \quad \text{for} \quad (x,y) \in D_{m,n}. \tag{A 2}$$

When $m = N$ or $n = N$, we replace the right-hand side of (A 2) by zero. Introduce the notation

$$\sup_{D_{m,n}} f(x,y) = f_{m,n}.$$

Also define the 'maximum local error' $\Delta_{m,n}$:

$$\Delta_{m,n}(x,y) = \sup_{D_{m,n}} [f(x,y) - \bar{f}(x,y)] \leqslant f_{m,n} - f_{m+1,n+1}.$$

We then obtain

$$\|f - \bar{f}\| = \int_0^1 \int_0^1 |f - \bar{f}| \,\mathrm{d}x\,\mathrm{d}y \leqslant \sum_{m=1}^{N} \sum_{n=1}^{N} \Delta_{m,n} \delta^2 \leqslant \delta^2 \sum_{m=1}^{N} \sum_{n=1}^{N} (f_{m,n} - f_{m+1,n+1}).$$

If the terms of this sum are displayed on a grid, it is easily seen that the contributions from all interior points cancel. Since $f_{m+1,n+1}$ is zero when $m = N$ or $n = N$, we are left with only the $2N-1$ terms $f_{m,n}$ where $m = 1$ or $n = 1$. Using $f_{m,n} \leqslant 1$, we finally obtain

$$\|f - \bar{f}\| \leqslant \delta^2(2N-1) < 2\delta. \tag{A 3}$$

The second, trivial step is to show that $\bar{f}$ can be approximated by some φ_i. Clearly, φ_i can be chosen so that $|\bar{f}-\varphi_i| \leqslant \delta$ for all (x,y) in D. Thus,

$$\|\bar{f}-\varphi_i\| = \int_0^1 \int_0^1 |\bar{f}-\varphi_i| \,\mathrm{d}x\,\mathrm{d}y \leqslant \delta. \tag{A 4}$$

Choosing $\delta = \epsilon/3$, we obtain the desired result (A 1) from (A 3) and (A 4). This proof also shows that I is totally bounded in $L^2[D]$, since the L^2-norm is bounded by the L^1-norm, as pointed out in §3.

What has been shown in this Appendix is a special case of a theorem that states that the set of functions on D with variation bounded by some constant is compact in $L^1[D]$ (Corollary 5.3.4 in Ziemer 1989).

Appendix B. Proof that the energy is a continuous functional

In this Appendix, we always use the L^2-norm. To show that E is a continuous functional on $L^2[D]$, we first rewrite (3) as a sum of two scalar products:

$$E[q] = -\tfrac{1}{2}(Sy^2, q) - \tfrac{1}{2}(q, Gq), \tag{B 1}$$

where G is the Green's operator, with the kernel $g(\boldsymbol{r}_1, \boldsymbol{r}_2) = (1/2\pi)\ln(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$. Since G is symmetric, we have, using the Cauchy–Schwartz inequality,

$$|(\hat{q}, G\hat{q}) - (q_i, Gq_i)| = |(\hat{q}-q_i, G(\hat{q}+q_i))| \leqslant \|\hat{q}-q_i\|\,\|G(\hat{q}+q_i)\| \leqslant \|\hat{q}-q_i\|\,\|G\|\,\|\hat{q}+q_i\|. \tag{B 2}$$

The norm of the Green's operator, defined by

$$\|G\|^2 = \frac{1}{2\pi}\int_{D_1}\int_{D_2} |\ln(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)|^2 \,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2,$$

is here finite. Using (B 2), we obtain

$$|E[\hat{q}] - E[q_i]| \leqslant \tfrac{1}{2}|(Sy^2, \hat{q}-q_i)| + \tfrac{1}{2}|(\hat{q}, G\hat{q}) - (q_i, Gq_i)| \leqslant \tfrac{1}{2}(\|Sy^2\| + \|G\|\,\|\hat{q}+q_i\|)\,\|\hat{q}-q_i\|, \tag{B 3}$$

demonstrating that $E[q_i] \rightarrow E[\hat{q}]$ as $q_i \rightarrow \hat{q}$ in $L^2[D]$.

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